On the use of Differential Calculus in the Resolution of fractions *

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§403 The method to resolve any propounded fraction into simple fractions which we explained in the *Introductio*, even though it is per se simple enough, can nevertheless be simplified by means of differential calculus in such a way that often the same task is completed by a much shorter calculation. Especially, if the denominator of the fraction to be resolved was of indefinite degree, the application of the method explained before is impeded a lot in most cases, since one has to find all factors of the denominator explicitly. But especially in these cases the division of the denominator by an already found factor becomes too cumbersome. This operation, if the differential calculus is applied, can be avoided such that it is not necessary to know the other factor of the denominator which results, if the denominator is divided by the known one. For this the method to determine the value of the fraction whose denominator and numerator vanish in a certain case can be applied; therefore, we want to teach in this chapter, how by means of that method the resolution of fractions treated above can be rendered more convenient and tractable, and at the same time want to finish this book in which we explained the use of differential calculus in the Analysis.

§404 Therefore, if any fraction $\frac{P}{Q}$ was propounded whose numerator and denominator are polynomial functions of the variable quantity x, one at first

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§405 Therefore, except for the integral part, if the fraction $\frac{P}{Q}$ contains one, one has to find the simple fractions whose denominators are either binomials of this form $f+gx$ or trinomials of this kind $f+2x\cos\phi\cdot\sqrt{fg}+gxx$ or squares or cubes or higher powers of formulas of this kind. And these denominators will be all factors of the denominator Q such that any arbitrary factor of this denominator Q yields a simple fraction. If the denominator Q has the factor $f+gx$, from it a simple fraction of this kind will result

$$\frac{\mathfrak{A}}{f+gx},$$

but if the factor was $(f+gx)^2$, the two factors

$$\frac{\mathfrak{A}}{(f+gx)^2}+\frac{\mathfrak{B}}{f+gx}.$$

And from a cubic factor $(f+gx)^3$ of the denominator Q three simple factors of this form will result

$$\frac{\mathfrak{A}}{(f+gx)^3}+\frac{\mathfrak{B}}{(f+gx)^2}+\frac{\mathfrak{C}}{f+gx}$$

and so forth. But if the denominator Q had a trinomial factor of this kind $ff-2fgx\cos\phi+ggxx$, from it a simple function of such a form will result

$$\frac{\mathfrak{A}+\mathfrak{a}x}{ff-2fgx\cos\phi+ggxx},$$

and if two factors of this kind were equal so that we have the factor $(ff-2fgx\cos\phi+ggxx)^2$, hence these two fractions will result

$$\frac{\mathfrak{A}+\mathfrak{a}x}{(ff-2fgx\cos\varphi+ggxx)^2}+\frac{\mathfrak{B}+\mathfrak{b}x}{ff-2fgx\cos\varphi+ggxx}.$$

But a cubic factor of this kind $(ff-2fgx\cos\phi+ggxx)^3$ will give three simple fractions, a fourth power four and so forth.

§406 Therefore, in order to resolve any fraction $\frac{P}{Q}$ proceed as follows. At first, find all simple either binomial or trinomial factors of the denominator Q, and if they were equal to each other, consider them as a single one. Then from these single factors of the denominator find the simple fractions either by the method already shown above or the one we will explain here and which can be applied instead of the first. Having done this the aggregate of all these fractions together with the integral part, if the propounded fraction $\frac{P}{Q}$ contains one, will be equal to the value of the fraction. We assume the factors of the denominator Q to be known here, since its depends on the resolution of the equation $Q=0$, and we will describe the method here to define the simple fraction resulting from any given factor of the denominator by means of differential calculus. This, because one already has the denominators of these simple fractions, will be achieved, if we teach to investigate the numerator of each fraction.

§407 Therefore, let us put that the denominator Q of the fraction $\frac{P}{Q}$ has the factor $f+gx$ such that it is $Q=(f+gx)S$ and this other factor S does not contain the same factor $f+gx$. Let the simple fraction resulting from this factor be $=\frac{\mathfrak{A}}{f+gx}$ and the complement will have a form of this kind $\frac{V}{S}$ such that it is

$$\frac{\mathfrak{A}}{f+gx}+\frac{V}{S}=\frac{P}{Q}.$$

Therefore, it will be

$$\frac{V}{S}=\frac{P}{Q}-\frac{\mathfrak{A}}{f+gx}=\frac{P-\mathfrak{A}S}{(f+gx)S}$$

and hence

$$V=\frac{P-\mathfrak{A}S}{f+gx}.$$

Therefore, since V is a polynomial function of x, it is necessary that $P - \mathfrak{A}S$ is divisible by $f + gx$; and therefore, if one puts $f + gx = 0$ or $x = \frac{-f}{g}$, the expression $P - \mathfrak{A}S$ will vanish. Therefore, let $x = \frac{-f}{g}$, and because it is $P - \mathfrak{A}S = 0$, it will be $\mathfrak{A} = \frac{P}{S}$, as we found already above. But because it is $S = \frac{Q}{f+gx}$, it will be

$$\mathfrak{A} = \frac{(f+gx)P}{Q},$$

if one puts $f + gx = 0$ or $x = \frac{-f}{g}$ everywhere. Because in this case so the numerator $(f + gx)P$ as the denominator Q vanish, by means of the things, which we explained on the investigation of the value of fractions of this kind, it will be

$$\mathfrak{A} = \frac{(f+gx)dP + Pgdx}{dQ},$$

if one puts $x = \frac{-f}{g}$. In this case because of $(f + gx)dP = 0$ it will be

$$\mathfrak{A} = \frac{gPdx}{dQ}$$

and so the value of the numerator $\mathfrak{A}$ will be found conveniently by means of differentiation.

§408 Therefore, if the denominator Q of the propounded fraction $\frac{P}{Q}$ has the simple factor $f + gx$, from it this simple fraction will result

$$\frac{\mathfrak{A}}{f+gx}$$

while $\mathfrak{A} = \frac{gPdx}{dQ}$, after here the value $\frac{-f}{g}$ that has to result from $f + gx = 0$ was substituted for x everywhere. Therefore, this way it is not necessary that one finds the other factor S of the denominator Q, which factor results, if Q is divided by $f + gx$. Hence, if Q is not expressed in factors, we can often omit this rather cumbersome division, especially if x has indefinite exponents in the denominator Q, since the value of A is obtained from the formula $\frac{gPdx}{dQ}$. But if the denominator Q was already expressed in factors such that hence the value of S is immediately plain, then rather the other expression should applied, by means of which we found $\mathfrak{A} = \frac{P}{S}$ by putting $x = \frac{-f}{g}$ everywhere in the same way. And so in any case one can apply that formula to find the value of $\mathfrak{A}$ which seems to more convenient. But we will illustrate the application of the new formula in some examples.

EXAMPLE 1

Let this fraction be propounded $\frac{x^9}{1+x^{17}}$ *whose simple fraction resulting from the factor* $1+x$ *is to be defined.*

Since here it is $Q = 1 + x^{17}$, even if its factor $1 + x$ is known, nevertheless, if, as the first method requires, we wanted to divide by it, it would result

$$S = 1 - x + xx - x^3 + \cdots + x^{16}.$$

Therefore, we will rather use the new formula $\mathfrak{A} = \frac{gPdx}{dQ}$; therefore, since it is $f = 1$, $g = 1$ and $P = x^9$, because of $dQ = 17x^{16}dx$ it will be $\mathfrak{A} = \frac{x^9}{17x^{16}} = \frac{1}{17x^7}$ for $x = -1$, whence it is $\mathfrak{A} = -\frac{1}{17}$, and the simple fraction resulting from the factor $1 + x$ of the denominator will be

$$\frac{-1}{17(1+x)}.$$

EXAMPLE 2

Having propounded the fraction $\frac{x^m}{1-x^{2n}}$ *to investigate the simple fraction resulting from the factor* $1 - x$.

Because of the propounded factor $1 - x$ it will be $f = 1$ and $g = -1$. But then the denominator $Q = 1 - x^{2n}$ gives $dQ = -2nx^{2n-1}dx$, whence because of $P = x^m$ one will obtain $\mathfrak{A} = \frac{-x^m}{-2nx^{2n-1}}$. And, from the equation $1 - x = 0$, having put $x = 1$ it will be $\mathfrak{A} = \frac{1}{2n}$ such that the simple fraction will be

$$\frac{1}{2n(1-x)}.$$

EXAMPLE 3

Having propounded the fraction $\frac{x^m}{1-4x^k+3x^n}$ *to determine the simple fraction resulting from the factor* $1 - x$.

Therefore, here it is $f = 1$ and $g = -1$, $P = x^m$, $Q = 1 - 4x^k + 3x^n$ and

$\frac{dQ}{dx} = -4kx^{k-1} + 3nx^{n-1}$; therefore, it its $\mathfrak{A} = \frac{-x^m}{-4kx^{k-1}+3nx^{n-1}}$ and for $x = 1$ it will be $\mathfrak{A} = \frac{1}{4k-3n}$. Therefore, the simple fraction resulting from this simple factor $1 - x$ of the denominator will be

$$\frac{1}{(4k-3n)(1-x)}.$$

§409 Let us now put that the denominator Q of the fraction $\frac{P}{Q}$ has the quadratic factor $(f+gx)^2$ and the simple fraction resulting from this is

$$\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}.$$

Let $Q = (f+gx)^2 S$ and the complement $= \frac{V}{S}$ such that it is

$$\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{(f+gx)^2} - \frac{\mathfrak{B}}{f+gx} \quad \text{and} \quad V = \frac{P - \mathfrak{A}S - \mathfrak{B}(f+gx)S}{(f+gx)^2}.$$

Since now V is an integer function, it is necessary that $P - \mathfrak{A}S - \mathfrak{B}S(f+gx)$ is divisible by $(f+gx)^2$; and because S does not further contain the factor $f+gx$, also this expression $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx)$ will be divisible by $(f+gx)^2$ and hence having put $f+gx = 0$ or $x = \frac{-f}{g}$ not only itself but also its differential $d.\frac{P}{S} - \mathfrak{B}gdx$ will vanish. Therefore, let $x = \frac{-f}{g}$ and from the first equation it will be $\mathfrak{A} = \frac{P}{S}$, from the second on the other hand it will be $\mathfrak{B} = \frac{1}{gdx}d.\frac{P}{S}$; having found these values one will have the fractions in question; for, it is

$$\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}.$$

Example

Having propounded the fraction $\frac{x^m}{1-4x^3+3x^4}$ whose numerator has the factor $(1-x)^2$, to find the simple fraction to result from it.

Since here it is $f = 1$, $g = -1$, $P = x^m$ and $Q = 1 - 4x^3 + 3x^4$, it will be $S = 1 - 2x + 3xx$,

$$\frac{P}{S} = \frac{x^m}{1+2x+3xx} \quad \text{and} \quad d.\frac{P}{S} = \frac{mx^{m-1}dx + 2(m-1)x^m dx + 3(m-2)x^{m+1}dx}{(1+2x+3xx)^2}.$$

Therefore, having put $x = 1$ it will be

$$\mathfrak{A} = \frac{1}{6} \quad \text{and} \quad \mathfrak{B} = -1 \cdot \frac{6m-8}{36} = \frac{4-3m}{18};$$

therefore, the fractions in question will be

$$\frac{1}{6(1-x)^2} + \frac{4-3m}{18(1-x)}.$$

§410 Let the denominator Q of the fraction $\frac{P}{Q}$ have three equal simple factors or let $Q = (f+gx)^3 S$ and let the simple fractions to result from this cubic factor $(f+gx)^3$ be these

$$\frac{\mathfrak{A}}{(f+gx)^3} + \frac{\mathfrak{B}}{(f+gx)^2} + \frac{\mathfrak{C}}{f+gx};$$

but let the complement of these fractions necessary to constitute the propounded fraction $\frac{P}{Q}$ be $\frac{V}{S}$ and it will be

$$V = \frac{P - \mathfrak{A}S - \mathfrak{B}S(f+gx) - \mathfrak{C}S(f+gx)^2}{(f+gx)^3}.$$

Hence this expression $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2$ will be divisible by $(f+gx)^3$; therefore, having put $f + gx = 0$ or $x = \frac{-f}{g}$ not only this expression itself but also its first and second differential will become $= 0$. By putting $x = \frac{-f}{g}$ it will be

$$\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2 = 0$$

$$d.\frac{P}{S} - \mathfrak{B}gdx - 2\mathfrak{C}gdx(f+gx) = 0$$

$$dd.\frac{P}{S} - 2\mathfrak{C}g^2dx^2 = 0.$$

From the first equation it will therefore be

$$\mathfrak{A} = \frac{P}{S}.$$

From the second on the other hand it will be

$$\mathfrak{B} = \frac{1}{gdx}d.\frac{P}{S}.$$

Finally, from the third one defines

$$\mathfrak{C} = \frac{1}{2g^2dx^2}dd.\frac{P}{S}.$$

§411 Therefore, in general, if the denominator Q of the fraction $\frac{P}{Q}$ has the factor $(f+gx)^n$ such that it is $Q = (f+gx)^nS$, having put the simple fractions to result from this factor $(f+gx)^n$

$$\frac{\mathfrak{A}}{(f+gx)^n} + \frac{\mathfrak{B}}{(f+gx)^{n-1}} + \frac{\mathfrak{C}}{(f+gx)^{n-2}} + \frac{\mathfrak{D}}{(f+gx)^{n-3}} + \frac{\mathfrak{A}}{(f+gx)^{n-4}} + \text{etc.},$$

until the last whose denominator is $f+gx$ is reached, if one reasons exactly as before, one will find that this expression

$$\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2 - \mathfrak{D}(f+gx)^3 - \mathfrak{E}(f+gx)^4 - \text{etc.}$$

must be divisible by $(f+gx)^n$; therefore, so itself as its single differentials up to degree $n-1$ will have to vanish in the case $x = \frac{-f}{g}$. From these equations one concludes that by putting $x = \frac{-f}{g}$ everywhere it will be

$$\begin{aligned}
\mathfrak{A} &= \frac{P}{S} \\
\mathfrak{B} &= \frac{1}{1gdx}d.\frac{P}{S} \\
\mathfrak{C} &= \frac{1}{1\cdot 2g^2dx^2}dd.\frac{P}{S} \\
\mathfrak{D} &= \frac{1}{1\cdot 2\cdot 3g^3dx^3}d^3.\frac{P}{S} \\
\mathfrak{E} &= \frac{1}{1\cdot 2\cdot 3\cdot 4g^4dx^4}d^4.\frac{P}{S} \\
&\text{etc.}
\end{aligned}$$

Here it is to be noted that these differentials of $\frac{P}{S}$ must be taken before one substitutes $\frac{-f}{g}$ for x; for, otherwise the variability of x would be lost.

§412 Therefore, in this way these numerators $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. will be expressed more easily than by the method given in the *Introductio* and often their values are also found a lot faster by means of this new method. In order to make this comparison more easily, let us define the values of the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. by means of the first method.

	Having put $x = \dfrac{-f}{g}$	Still considering x as a variable, set
it will be	$\mathfrak{A} = \dfrac{P}{S}$	$\dfrac{P - \mathfrak{A}S}{f + gx} = \mathfrak{P}$,
it will be	$\mathfrak{B} = \dfrac{\mathfrak{P}}{S}$	$\dfrac{\mathfrak{P} - \mathfrak{B}S}{f + gx} = \mathfrak{Q}$,
it will be	$\mathfrak{C} = \dfrac{\mathfrak{Q}}{S}$	$\dfrac{\mathfrak{Q} - \mathfrak{C}S}{f + gx} = \mathfrak{R}$,
it will be	$\mathfrak{D} = \dfrac{\mathfrak{R}}{S}$	$\dfrac{\mathfrak{R} - \mathfrak{D}S}{f + gx} = \mathfrak{S}$,
it will be	$\mathfrak{E} = \dfrac{\mathfrak{S}}{S}$	and so forth.

§413 But if the denominator Q of the fraction $\frac{P}{Q}$ has not only simple real factors, then take two imaginary ones whose product will be real then. Therefore, let the factor of the denominator Q be $ff - 2fgx\cos\varphi + ggxx$ which put $= 0$ gives these two imaginary values

$$x = \frac{f}{g}\cos\varphi \pm \frac{f}{g\sqrt{-1}}\sin\varphi;$$

therefore, it will be

$$x^n = \frac{f^n}{g^n}\cos n\varphi \pm \frac{f^n}{g^n\sqrt{-1}}\sin n\varphi.$$

Let us put that it is $Q = (ff - 2fgx\cos\varphi + ggxx)S$ and additionally S is not divisible by $ff - 2fgx\cos\varphi + ggxx$. Let the fraction to result from this factor be

$$\frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\varphi + ggxx}$$

and let the complement necessary to get to the propounded fraction $\frac{P}{Q}$ be $= \frac{V}{S}$; it will be

$$V = \frac{P - (\mathfrak{A} + \mathfrak{a}x)S}{ff - 2fgx\cos\varphi + ggxx},$$

whence $P - (\mathfrak{A} + \mathfrak{a}x)S$ and therefore also $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ will be divisible by $ff - 2fgx\cos\varphi + ggxx = 0$, this means, if one puts either

$$x = \frac{f}{g}\cos\varphi + \frac{f}{g\sqrt{-1}}\sin\varphi$$

or

$$x = \frac{f}{g}\cos\varphi - \frac{f}{g\sqrt{-1}}\sin\varphi.$$

§414 Since P and S are polynomial functions of x, make both substitutions separately in both expressions; and since for any power of x, say x^n, this binomial

$$x^n = \frac{f^n}{g^n}\cos n\varphi \pm \frac{f^n}{g^n\sqrt{-1}}\sin n\varphi$$

has to be substituted, first let us put $\frac{f^n}{g^n}\cos n\varphi$ for x^n everywhere and having done this let P go over into $\mathfrak{P}$ and S into $\mathfrak{S}$. Further, put $\frac{f^n}{g^n}\sin n\varphi$ for x^n everywhere and having done this let P go over into $\mathfrak{p}$ and S into $\mathfrak{s}$; here it is to be noted that before these substitutions both functions P and S have to be expanded completely such that, if they are contained in factors, one has to get rid of these factors by actual multiplication. Having found these values $\mathfrak{P}$, $\mathfrak{p}$, $\mathfrak{S}$, $\mathfrak{s}$ it will be obvious, if one puts $x = \frac{f}{g}\cos\varphi \pm \frac{f}{g\sqrt{-1}}\sin\varphi$, that the function P will go over into $\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}}$ and the function S will go over into $\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$. Therefore, because $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ or $P - (\mathfrak{A} + \mathfrak{a}x)S$ has to vanish in both cases, it will be

$$\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}} = \left(\mathfrak{A} + \frac{\mathfrak{a}f}{g}\cos\varphi \pm \frac{\mathfrak{a}f}{g\sqrt{-1}}\sin\varphi\right)\left(\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}\right),$$

whence because of the ambiguous signs these two equations will result

$$\mathfrak{P} = \mathfrak{A}\mathfrak{S} + \frac{\mathfrak{a}f\mathfrak{S}}{g}\cos\varphi - \frac{\mathfrak{a}f\mathfrak{s}}{\sin\varphi},$$
$$\mathfrak{p} = \mathfrak{A}\mathfrak{s} + \frac{\mathfrak{a}f\mathfrak{s}}{g}\cos\varphi - \frac{\mathfrak{a}f\mathfrak{S}}{\sin\varphi},$$

from which by eliminating $\mathfrak{A}$ one finds

$$\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P} = \frac{\mathfrak{a}f(\mathfrak{S}^2 + \mathfrak{s}^2)}{g}\sin\varphi;$$

and hence it will be

$$\mathfrak{a} = \frac{g(\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P})}{f(\mathfrak{S}^2 + \mathfrak{s}^2)\sin\varphi}.$$

Further, by eliminating $\sin\varphi$ it will be

$$\mathfrak{S}\mathfrak{P} + \mathfrak{s}\mathfrak{p} = (\mathfrak{S}^2 + \mathfrak{s}^2)(\mathfrak{A} + \frac{\mathfrak{a}f}{g}\cos\varphi).$$

Therefore,

$$\mathfrak{A} = \frac{\mathfrak{S}\mathfrak{P} + \mathfrak{s}\mathfrak{p}}{\mathfrak{S}^2 + \mathfrak{s}^2} - \frac{(\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P}\cos\varphi)}{(\mathfrak{S}^2 + \mathfrak{s}^2)\sin\varphi}.$$

§415 Since it is

$$S = \frac{Q}{ff - 2fgx\cos\varphi + ggxx}$$

and since having put

$$ff - 2fgx\cos\varphi + ggxx = 0$$

so the numerator as the denominator will vanish, in this case it will be

$$S = \frac{dQ : dx}{2ggx - 2fg\cos\varphi}.$$

Now, let us put, if one substitutes $x^n = \frac{f^n}{g^n}\cos n\varphi$ everywhere, that the function $\frac{dQ}{dx}$ goes over into $\mathfrak{Q}$, but if one sets $x^n = \frac{f^n}{g^n}\sin n\varphi$, that it goes over into $\mathfrak{q}$;

and it is obvious, if one puts $x = \frac{f}{g}\cos\varphi \pm \frac{f}{g\sqrt{-1}}\sin\varphi$, that the function $\frac{dQ}{dx}$ goes over into $\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}}$. From this the function S will go over into

$$\frac{\mathfrak{Q} \pm \mathfrak{q} : \sqrt{-1}}{\pm 2fg\sin\varphi : \sqrt{-1}}.$$

Therefore, since it is $S = \mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$ having substituted the same value for x, one will have

$$\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}} = \pm\frac{2fg\mathfrak{S}}{\sqrt{-1}}\sin\varphi - 2fg\mathfrak{s}\sin\varphi.$$

Therefore, it will be

$$\mathfrak{s} = \frac{-\mathfrak{Q}}{2fg\sin\varphi} \quad \text{and} \quad \mathfrak{S} = \frac{\mathfrak{q}}{2fg\sin\varphi}.$$

And having substituted these values it will be

$$\mathfrak{a} = \frac{2gg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}$$

and

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\varphi}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})\cos\varphi}{\mathfrak{Q}^2 + \mathfrak{q}^2}.$$

§416 Therefore, we found an appropriate way to form the simple fraction from each factor of second power and here, since the denominator itself of the propounded fraction is retained in the calculation, we avoid the division, by means of which the value of the letter S would have to be defined and which is often very cumbersome. Therefore, if the denominator Q of the fraction $\frac{P}{Q}$ has such a factor $ff - 2fgx\cos\varphi + ggxx$, the simple fraction to result from this factor and we assumed to be

$$= \frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\varphi + ggxx}$$

will be defined the following way. Put $x = \frac{f}{g}\cos\varphi$ and for each power x^n of x write $\frac{f^n}{g^n}\cos n\varphi$; having done this let P go over into $\mathfrak{P}$ and the function $\frac{dQ}{dx}$ into $\mathfrak{Q}$. Further, put $x = \frac{f}{g}\sin\varphi$ and each of its powers $x^n = \frac{f^n}{g^n}\sin n\varphi$ and let P go over into $\mathfrak{p}$ and $\frac{dQ}{dx}$ into $\mathfrak{q}$. And, having found these values of the letters $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{p}$ and $\mathfrak{q}$ this way the quantities $\mathfrak{A}$ and $\mathfrak{a}$ will be defined in such a way that it is

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q}\sin\varphi)}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})\cos\varphi}{\mathfrak{Q}^2 + \mathfrak{q}^2},$$

$$\mathfrak{a} = \frac{2gg(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}.$$

Therefore, the fraction to result from the factor $ff - 2fg\cos\varphi + ggxx$ of the denominator Q will be

$$\frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\varphi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})(gx - f\cos\varphi)}{(\mathfrak{Q}^2 + \mathfrak{q}^2)(ff - 2fgx\cos\varphi + ggxx)}.$$

EXAMPLE 1

If this fraction $\frac{x^m}{a+bx^n}$ was propounded whose denominator $a + bx^n$ has this factor $ff - 2fgx\cos\varphi + ggxx$, to find the simple fraction corresponding to this factor.

Since here it is $P = x^m$ and $q = a + bx^n$, it will be

$$\frac{dQ}{dx} = nbx^{n-1},$$

whence it will be

$$\mathfrak{P} = \frac{f^m}{g^m}\cos m\varphi, \qquad \mathfrak{p} = \frac{f^m}{g^m}\sin m\varphi,$$

$$\mathfrak{Q} = \frac{nbf^{n-1}}{g^{n-1}}\cos(n-1)\varphi, \quad \mathfrak{q} = \frac{nbf^{n-1}}{g^{n-1}}\sin(n-1)\varphi.$$

From these it will be

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2b^2f^{2(n-1)}}{g^{2(n-1)}},$$

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\sin(n-m-1)\varphi$$

and

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}}\cos(n-m-1)\varphi.$$

Therefore, the simple fraction in question will be

$$\frac{2g^{n-m}(f\sin\varphi\cdot\sin(n-m-1)\varphi+gx\cos(n-m-1)\varphi-f\cos\varphi\cdot\cos(n-m-1)\varphi)}{nbf^{n-m-1}(ff-2fgx\cos\varphi+ggxx)}$$

or

$$\frac{2g^{n-m}(gx\cos(n-m-1)\varphi-f\cos(n-m)\varphi)}{nbf^{n-m-1}(ff-2fgx\cos\varphi+ggxx)}.$$

EXAMPLE 2

Let this fraction $\frac{1}{x^m(a+bx^n)}$ be propounded whose denominator has the factor $ff-2fgx\cos\varphi+ggxx$; to find the simple fraction to result from this.

Since it is $P=1$ and $Q=ax^m+bx^{m+n}$, it will be

$$\frac{dQ}{dx}=max^{m-1}+(m+n)bx^{m+n-1}$$

and hence having put $x^n=\frac{f^n}{g^n}\cos n\varphi$ because of $P=x^0$, $\mathfrak{P}=1$ it will be

$$\mathfrak{Q}=\frac{maf^{m-1}}{g^{m-1}}\cos(m-1)\varphi+\frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\cos(m+n-1)\varphi$$

and, having put $x^n=\frac{f^n}{g^n}\sin n\varphi$, it will be $\mathfrak{p}=0$ and

$$\mathfrak{q}=\frac{maf^{m-1}}{g^{m-1}}\sin(m-1)\varphi+\frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\sin(m+n-1)\varphi.$$

Therefore,

$$\mathfrak{Q}^2+\mathfrak{q}^2=\frac{m^2a^2f^{2(m-1)}}{g^2(m-1)}+\frac{2m(m+n)abf^{2m+n-2}}{g^{2m+n-2}}\cos n\varphi+\frac{(m+n)^2b^2f^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

If $ff-2fgx\cos\varphi+ggxx$ is a divisor of $a+bx^n$, it will be

$$a+\frac{bf^n}{g^n}\cos n\varphi=0\quad\text{and}\quad\frac{bf^n}{g^n}\sin n\varphi=0,\quad\text{whence}\quad aa=\frac{bbf^{2n}}{g^{2n}}.$$

Therefore, it will be

$$\mathfrak{Q}^2+\mathfrak{q}^2=\frac{(m+n)^2bbf^{2(m+n-1)}}{g^{2(m+n-1)}}-\frac{m(2n+m)aaf^{2(m-1)}}{g^{2(m-1)}}=\frac{nnaaf^{2(m-1)}}{g^{2(m-1)}}=\frac{nnbbf^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

Further, it will be

$$\begin{aligned}\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}&=\frac{maf^{m-1}}{g^{m-1}}\sin(m-1)\varphi+\frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\sin(m+n-1)\varphi\\&=\frac{bf^{m+n-1}}{g^{m+n-1}}((m+n)\sin(m+n-1)\varphi-m\cos n\varphi\cdot\sin(m-1)\varphi)\\&=\frac{bf^{m+n-1}}{g^{m+n-1}}(n\cos n\varphi\cdot\sin(m-1)\varphi+(m+n)\sin n\varphi\cdot\cos(m-1)\varphi)\end{aligned}$$

and

$$\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}=\frac{bf^{m+n-1}}{g^{m+n-1}}((m+n)\cos(m+n-1)\varphi-m\cos n\varphi\cdot\cos(m-1)\varphi).$$

Or because $ff-2fg\cos\varphi+ggxx$ is also a divisor of $ax^{m-1}+bx^{m+n-1}$, it will be

$$\frac{af^{m-1}}{g^{m-1}}\cos(m-1)\varphi+\frac{bf^{m+n-1}}{g^{m+n-1}}\cos(m+n-1)\varphi=0$$

and

$$\frac{af^{m-1}}{g^{m-1}}\sin(m-1)\varphi+\frac{bf^{m+n-1}}{g^{m+n-1}}\sin(m+n-1)\varphi=0,$$

whence it will be

$$\mathfrak{Q}=\frac{nbf^{m+n-1}}{g^{m+n-1}}\cos(m+n-1)\varphi\quad\text{and}\quad\mathfrak{q}=\frac{nbf^{m+n-1}}{g^{m+n-1}}\sin(m+n-1)\varphi$$

or

$$\mathfrak{Q}=\frac{-naf^{m-1}}{g^{m-1}}\cos(m-1)\varphi\quad\text{and}\quad\mathfrak{q}=\frac{-naf^{m-1}}{g^{m-1}}\cos(m-1)\varphi.$$

From these the fraction in question will result as

$$\frac{2g^m(f\cos m\varphi - gx\cos(m-1)\varphi)}{naf^{m-1}(ff - 2fgx\cos\varphi + ggxx)}.$$

This formula formula follows from the first example, if one substitutes a negative m, whence it would not have been necessary to consider this case.

EXAMPLE 3

If the denominator of this fraction $\frac{x^m}{a+bx^n+cx^{2n}}$ *has the factor* $ff - 2fgx\cos\varphi + ggxx$*, to investigate the simple fraction to result from this factor.*

If $ff - 2fgx\cos\varphi + ggxx$ is a factor of the denominator $a + bx^n + cx^{2n}$, it will, as we showed above, be

$$a + \frac{bf^n}{g^n}\cos n\varphi + \frac{cf^{2n}}{g^{2n}} = 0 \quad \text{and} \quad \frac{bf^n}{g^n}\sin n\varphi + \frac{cf^{2n}}{g^{2n}}\sin 2n\varphi = 0.$$

Therefore, since it is $P = x^m$ and $Q = a + bx^n + cx^{2n}$, it will be

$$\frac{dQ}{dx} = nbx^{n-1} + 2ncx^{2n-1},$$

whence it is

$$\mathfrak{P} = \frac{f^m}{g^m}\cos m\varphi \quad \text{and} \quad \mathfrak{p} = \frac{f^m}{g^m}\sin m\varphi,$$

$$\mathfrak{Q} = \frac{nbf^{n-1}}{g^{n-1}}\cos(n-1)\varphi + \frac{2ncf^{2n-1}}{g^{2n-1}}\cos(2n-1)\varphi,$$
$$\mathfrak{q} \;= \frac{nbf^{n-1}}{g^{n-1}}\sin(n-1)\varphi + \frac{2ncf^{2n-1}}{g^{2n-1}}\sin(2n-1)\varphi,$$

Therefore, we will have

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2 f^{2(n-1)}}{g^{2(n-1)}}\left(bb + \frac{4bcf^n}{g^n}\cos n\varphi + \frac{4ccf^{2n}}{g^{2n}}\right).$$

But from the first two equations it is

$$\frac{f^{2n}}{g^{2n}}\left(bb+\frac{2bcf^n}{g^n}\cos n\varphi+\frac{ccf^{2n}}{g^{2n}}\right)=aa$$

and hence

$$\frac{4bcf^n}{g^n}\cos n\varphi=\frac{2g^{2n}aa}{f^{2n}}-2bb-\frac{2ccf^{2n}}{g^{2n}};$$

having substituted the value there it will be

$$\mathfrak{Q}^2+\mathfrak{q}^2=\frac{n^2f^{2n-2}}{g^{2n-2}}\left(\frac{2aag^{2n}}{f^{2n}}-bb+\frac{2ccf^{2n}}{g^{2n}}\right)$$

or

$$\mathfrak{Q}^2+\mathfrak{q}^2=\frac{n^2(2aag^{4n}-bbf^{2n}g^{2n}+2ccf^{4n})}{ffg^{4n-2}}.$$

Further, it will be

$$\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}=\frac{nbf^{m+n-1}}{g^{m+n-1}}\sin(n-m-1)\varphi+\frac{2ncf^{m+2n-1}}{g^{m+2n-1}}\sin(2n-m-1)\varphi,$$

$$\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}=\frac{nbf^{m+n-1}}{g^{m+n-1}}\cos(n-m-1)\varphi+\frac{2ncf^{m+2n-1}}{g^{m+2n-1}}\cos(2n-m-1)\varphi.$$

Having found these values the simple fraction in question will be

$$\frac{2fg(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q})\sin\varphi+2g(\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q})(gx-f\cos\varphi)}{(\mathfrak{Q}^2+\mathfrak{q}^2)(ff-2fgx\cos\varphi+ggxx)}.$$

§417 But these fractions will be expressed in an easier way, if we determine the factors of the denominators. Therefore, let the denominator of the propounded fraction be

$$a+bx^n;$$

if the trinomial factor is put

$$ff-2fgx\cos\varphi+ggxx,$$

it will be, as we showed in the introduction,

$$a+\frac{bf^n}{g^n}\cos n\varphi=0 \quad \text{and} \quad \frac{bf^n}{g^n}\sin n\varphi=0;$$

therefore, because it is $\sin n\varphi=0$, it will be either $n\varphi=(2k-1)\pi$ or $n\pi=2k\pi$; in the first case it will be $\cos n\varphi=-1$, in the second $\cos n\varphi=+1$. Therefore, if a and b are positive quantities, only the first case will hold, in which it is $a=\frac{bf^n}{g^n}$ and therefore

$$f=a^{\frac{1}{n}} \quad \text{and} \quad g=b^{\frac{1}{n}}.$$

But instead of these irrational quantities let us still use the letters f and g or let us put $a=f^n$ and $b=g^n$ such the the factors of this function have to be investigated

$$f^n+g^nx^n.$$

Therefore, since it is $\varphi=\frac{(2k-1)\pi}{n}$, where k can denote any positive integer, but on the other hand for k no larger numbers as those which render $\frac{2k-1}{n}$ greater than 1 are to be taken; therefore, the factors of the propounded fraction $f^n+g^nx^n$ will be the following

$$\begin{gathered}
ff-2fgx\cos\frac{\pi}{n}+ggxx\\
ff-2fgx\cos\frac{3\pi}{n}+ggxx\\
ff-2fgx\cos\frac{5\pi}{n}+ggxx\\
\text{etc.},
\end{gathered}$$

where it is to be noted, if n is an odd number, that one has this one binomial factor

$$f+gx;$$

but if n is an even number, the product will contain no binomial factor.

Example 1

To resolve this fraction $\frac{x^m}{f^n+g^nx^n}$ into its simple fractions.

Since a trinomial factor of any arbitrary denominator is contained in this form

$$ff - 2fgx\cos\frac{(2k-1)\pi}{n} + ggxx,$$

in example 1 of the preceding paragraph it will be $a = f^n$, $b = g^n$ and $\varphi = \frac{(2k-1)\pi}{n}$, whence it will be

$$\sin(n-m-1)\varphi = \sin(m+1)\varphi = \sin\frac{(m+1)(2k-1)\pi}{n}$$

and

$$\cos(n-m-1)\varphi = -\cos(m+1)\varphi = -\cos\frac{(m+1)(2k-1)\pi}{n}.$$

Therefore, from this factor this simple fraction results

$$\frac{2f\sin\frac{(2k-1)\pi}{n}\cdot\sin\frac{(m+1)(2k-1)\pi}{n} - 2\cos\frac{(m+1)(2k-1)\pi}{n}\left(fx - f\cos\frac{(2k-1)\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{(2k-1)\pi}{n} + ggxx\right)}.$$

Therefore, the propounded fraction will be resolved into these simple ones

$$\begin{gathered}\frac{2f\sin\frac{\pi}{n}\cdot\sin\frac{(m+1)\pi}{n} - 2\cos\frac{(m+1)\pi}{n}\left(fx - f\cos\frac{\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{\pi}{n} + ggxx\right)}\\
+\frac{2f\sin\frac{3\pi}{n}\cdot\sin\frac{3(m+1)\pi}{n} - 2\cos\frac{3(m+1)\pi}{n}\left(fx - f\cos\frac{3\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{3\pi}{n} + ggxx\right)}\\
+\frac{2f\sin\frac{5\pi}{n}\cdot\sin\frac{5(m+1)\pi}{n} - 2\cos\frac{5(m+1)\pi}{n}\left(fx - f\cos\frac{5\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{5\pi}{n} + ggxx\right)}\\
\text{etc.}\end{gathered}$$

Therefore, if n was an even number, this way all simple fractions result; but if n was an odd number, because of the binomial factor $f + gx$ to the fractions resulting this way one furthermore has to add this one

$$\frac{\pm}{nf^{n-m-1}g^m(f+gx)},$$

where the sign $+$ holds, if m was an even number, otherwise the sign $-$. If m was a number greater than m, then to these fractions additionally integral parts of this kind are added

$$Ax^{m-n} + Bx^{m-2n} + C^{m-3n} + Dx^{m-4n} + \text{etc.},$$

as long as the exponents remain positive, and it will be

$$\begin{aligned}
Ag^n &= 1 \quad \text{therefore} & A &= +\frac{1}{g^n} \\
Af^n + Bg^n &= 0 & B &= -\frac{f^n}{g^{2n}} \\
Bf^n + Cg^n &= 0 & C &= +\frac{f^{2n}}{g^{3n}} \\
Cf^n + Dg^n &= 0 & C &= -\frac{f^{3n}}{g^{4n}} \\
&\text{etc.} & &\text{etc.}
\end{aligned}$$

EXAMPLE 2

To resolve this fraction $\frac{1}{x^m(f^n+g^nx^n)}$ *into its simple fractions.*

Concerning the factors of $f^n + g^nx^n$, from them the same fractions we found in the preceding example result, if only m is taken negatively; therefore, it only remains that we define the simple fractions resulting from the other factor x^m, which is most conveniently done this way. Set the propounded fraction

$$= \frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{N}x^{n-m}}{f^n + g^nx^n}$$

and it will be

$$\begin{aligned}
\mathfrak{A}\mathfrak{f}^{\mathfrak{n}} &= 1 \quad \text{therefore} & \mathfrak{A} &= +\frac{1}{f^n} \\
\mathfrak{A}g^n + \mathfrak{N} &= 0 & \mathfrak{N} &= -\frac{g^n}{f^n}.
\end{aligned}$$

If $n - m$ was a negative number, one will have to proceed in like manner, such that, if m was an arbitrary large number, simple fractions of this kind result

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \text{etc.},$$

of which series so many terms are to be taken as one has positive exponents of x in the denominator. And it will be

$$\begin{aligned}
\mathfrak{A}f^n &= 1 \quad \text{therefore} & \mathfrak{A} &= +\frac{1}{f^n}\\
\mathfrak{A}g^n + \mathfrak{B}g^n &= 0 & \mathfrak{B} &= -\frac{g^n}{f^{2n}}\\
\mathfrak{B}g^n + \mathfrak{C}g^n &= 0 & \mathfrak{C} &= -\frac{g^{2n}}{f^{3n}}\\
\mathfrak{C}g^n + \mathfrak{D}g^n &= 0 & \mathfrak{D} &= -\frac{g^{3n}}{f^{4n}}\\
&\text{etc.} & &\text{etc.}
\end{aligned}$$

Therefore, the propounded fraction in total will be resolved into these simple fractions

$$\frac{1}{f^n x^m} - \frac{g^n}{f^{2n}x^{m-n}} + \frac{g^{2n}}{f^{3n}x^{m-2n}} - \frac{g^{3n}}{f^{4n}x^{m-3m}} + \text{etc.}$$
$$-\frac{2fg^m \sin\frac{\pi}{n}\cdot\sin\frac{(m-1)\pi}{n} + 2g^m\cos\frac{(m-1)\pi}{n}\left(gx - f\cos\frac{\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{\pi}{n} + ggxx\right)}$$
$$-\frac{2fg^m \sin\frac{3\pi}{n}\cdot\sin\frac{3(m-1)\pi}{n} + 2g^m\cos\frac{3(m-1)\pi}{n}\left(gx - f\cos\frac{3\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{3\pi}{n} + ggxx\right)}$$
$$-\frac{2fg^m \sin\frac{5\pi}{n}\cdot\sin\frac{5(m-1)\pi}{n} + 2g^m\cos\frac{5(m-1)\pi}{n}\left(gx - f\cos\frac{5\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{5\pi}{n} + ggxx\right)}$$
$$\text{etc.}$$

To these formulas, if n was an odd number, because of the factor $f + gx$ of the denominator one furthermore has to add

$$\frac{\pm g^m}{nf^{n+m-1}(f+gx)},$$

where the upper of the two signs $\pm$ holds, if m was an even number, the lower on the other hand, if m was odd.

§418 Now, let us also consider the formula $a + bx^n$, if b was a negative number, and let this function be propounded

$$f^n - g^n x^n,$$

of which at first one factor will always be $f - gx$; and if n is an even number, also $f + gx$ will be a factor. The remaining factors on the other hand are trinomial factors; if their general form is put

$$ff - 2fgx\cos\varphi + ggxx,$$

it will be

$$f^n - f^n\cos n\varphi = 0 \quad \text{and} \quad f^n \sin n\varphi = 0$$

or

$$\sin n\varphi = 0 \quad \text{and} \quad \cos n\varphi = 1.$$

To satisfy those equations, it is necessary that it is $n\varphi = 2k\pi$ while k is any integer number and therefore it will be $\varphi = \frac{2k\pi}{n}$. Therefore, the general factor will be

$$ff - 2fgx\cos\frac{2k\pi}{n} + ggxx;$$

therefore, by taking even numbers smaller than the exponent n for $2k$ all trinomial factors will result

$$\begin{aligned}
&ff - 2fgx\cos\frac{2\pi}{n} + ggxx\\
&ff - 2fgx\cos\frac{4\pi}{n} + ggxx\\
&ff - 2fgx\cos\frac{6\pi}{n} + ggxx\\
&\text{etc.}
\end{aligned}$$

EXAMPLE 1

To resolve this fraction $\frac{x^m}{f^n - g^n x^n}$ *into its simple fractions.*

Since one factor of the denominator is $f - gx$, hence a fraction of this kind will result $\frac{\mathfrak{A}}{f-gx}$; to find its numerator, put $P = x^m$ and $f^n - g^n x^n = Q$; it will be

$$dQ = -ng^n x^{n-1} dx$$

and it will be

$$\mathfrak{A} = \frac{-gx^m}{-ng^n x^{n-1}} = \frac{x^m}{ng^{n-1}x^{n-1}}$$

having put $x = \frac{f}{g}$. Therefore, it will be $\mathfrak{A} = \frac{1}{nf^{n-m-1}g^m}$ and hence the simple fraction resulting from the factor $f - gx$ will be

$$\frac{1}{nf^{n-m-1}g^m(f-gx)}.$$

If n is an even number, since then one factor of the denominator also is $f + gx$, put the simple fraction to result from this $= \frac{\mathfrak{A}}{f+gx}$; it will be

$$\mathfrak{A} = \frac{-gx^m}{ng^n x^{n-1}} = \frac{-x^m}{ng^{n-1}x^{n-1}}$$

having put $x = \frac{-f}{g}$. Therefore, because of the odd number $n-1$ it will be $g^{n-1}x^{n-1} = -f^{n-1}$; but it will be $x^m = \frac{\pm f^m}{g^m}$, where the upper sign holds, if m was an even number, the lower, if m was an odd number. Therefore, because it is $\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1}g^m}$, the simple fraction to result from the factor $f + gx$ will be this

$$\frac{\mp 1}{nf^{n-m-1}g^m(f+gx)}.$$

Further, because the general form of trinomial factors is

$$ff - 2fgx\cos\frac{2k\pi}{n} + ggxx,$$

if we make the comparison to example 1 § 416, it will be $a = f^n$, $b = -g^n$ and $\varphi = \frac{2k\pi}{n}$; hence

$$\sin n\varphi = \quad 0\text{and} \quad \cos n\varphi = 1$$

and

$$\sin(n-m-1)\varphi = -\sin(m+1)\varphi = -\sin\frac{2k(m+1)\pi}{n}$$

and

$$\cos(n-m-1)\varphi = \cos(m+1)\varphi = \cos\frac{2k(m+1)\pi}{n}.$$

From these equations the simple fraction to result from this will be

$$\frac{2f\sin\frac{2k\pi}{n}\cdot\sin\frac{2k(m+1)\pi}{n} - 2\cos\frac{2k(m+1)\pi}{n}\left(gx - f\cos\frac{2k\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{2k\pi}{n} + ggxx\right)}.$$

Therefore, the simple fractions in question will be

$$\begin{aligned}
&\frac{1}{nf^{n-m-1}g^m(f-gx)}\\
&+\frac{2f\sin\frac{2\pi}{n}\cdot\sin\frac{2(m+1)\pi}{n} - 2\cos\frac{2(m+1)\pi}{n}\left(gx - f\cos\frac{2\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{2\pi}{n} + ggxx\right)}\\
&+\frac{2f\sin\frac{4\pi}{n}\cdot\sin\frac{4(m+1)\pi}{n} - 2\cos\frac{4(m+1)\pi}{n}\left(gx - f\cos\frac{4\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{4\pi}{n} + ggxx\right)}\\
&+\frac{2f\sin\frac{6\pi}{n}\cdot\sin\frac{6(m+1)\pi}{n} - 2\cos\frac{6(m+1)\pi}{n}\left(gx - f\cos\frac{6\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff - 2fgx\cos\frac{6\pi}{n} + ggxx\right)}\\
&\text{etc.,}
\end{aligned}$$

to which, if n was an even number, one furthermore has to add this fraction

$$\frac{\mp 1}{nf^{n-m-1}g^m(f+gx)},$$

of which the upper sign $-$ is to be taken, if m was an even number, the lower, if odd. Furthermore, if m is a number not smaller than n, this integer parts are to be added

$$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \text{etc.},$$

as long the exponents were not negative, and it will be

$$
\begin{aligned}
-Ag^n &= 1 \quad \text{therefore} & A &= -\frac{1}{g^n} \\
Af^n - Bg^n &= 0 & B &= -\frac{f^n}{g^{2n}} \\
Bf^n - Cg^n &= 0 & C &= -\frac{f^{2n}}{g^{3n}} \\
Cf^n - Dg^n &= 0 & C &= -\frac{f^{3n}}{g^{4n}} \\
\text{etc.} & & \text{etc.}
\end{aligned}
$$

EXAMPLE 2

To resolve this fraction $\frac{1}{x^m(f^n - g^n x^n)}$ *into its simple fractions.*

The fractions resulting from the factor $f^n - g^n x^n$ of the denominator will be the same as before, as long as in those formulas m is taken negatively. Hence one has to consider the other factor x^m; if we put that from this these fractions the following expression results

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \text{etc.},$$

which is series is to be continued until the exponents of x become negative, it will be

$$
\begin{aligned}
+\mathfrak{A}f^n &= 1 \quad \text{therefore} & \mathfrak{A} &= \frac{1}{f^n} \\
\mathfrak{B}f^n - \mathfrak{A}g^n &= 0 & \mathfrak{B} &= \frac{g^n}{f^{2n}} \\
\mathfrak{C}f^n - \mathfrak{B}g^n &= 0 & \mathfrak{C} &= \frac{g^{2n}}{f^{3n}} \\
\mathfrak{D}f^n - \mathfrak{C}g^n &= 0 & \mathfrak{D} &= \frac{g^{3n}}{f^{4n}} \\
\text{etc.} & & \text{etc.}
\end{aligned}
$$

Therefore, the propounded fraction will be resolved into these simple fractions

$$\frac{1}{f^n x^m} + \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} + \frac{g^{3n}}{f^{4n} x^{m-3n}} + \text{etc.}$$

$$+\frac{g^m}{nf^{n+m-1}(f-gx)}$$

$$-\frac{2fg^m \sin\frac{2\pi}{n} \cdot \sin\frac{2(m-1)\pi}{n} + 2g^m \cos\frac{2(m-1)\pi}{n}\left(gx - f\cos\frac{2\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{2\pi}{n} + ggxx\right)}$$

$$-\frac{2fg^m \sin\frac{4\pi}{n} \cdot \sin\frac{4(m-1)\pi}{n} + 2g^m \cos\frac{4(m-1)\pi}{n}\left(gx - f\cos\frac{4\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{4\pi}{n} + ggxx\right)}$$

$$-\frac{2fg^m \sin\frac{6\pi}{n} \cdot \sin\frac{6(m-1)\pi}{n} + 2g^m \cos\frac{6(m-1)\pi}{n}\left(gx - f\cos\frac{6\pi}{n}\right)}{nf^{n+m-1}\left(ff - 2fgx\cos\frac{6\pi}{n} + ggxx\right)}$$

$$\text{etc.},$$

to which, if n was an even number, one additionally has to add this fraction

$$\frac{\mp g^m}{nf^{n+m-1}(f+gx)},$$

which is omitted, if n was an odd number. The upper sign $-$ of the ambiguous signs holds, if m is an even number, the lower $+$ on the other hand, if m is an odd number.

§419 Therefore, this way all fractions whose denominator consists of two terms of this kind $a + bx^n$ are resolved into simple fractions. But if the denominator consists of three terms of this kind $a + bx^n + cx^{2n}$, then one has to see at first, whether it can be resolved into two real factors of this first form. For, if this is possible, the resolution into simple fractions can be done in the way explained before. For, if a fraction of this kind is propounded

$$\frac{x^m}{(f^n + g^n x^n)(f^n + k^n x^n)},$$

it will at first be transformed into two of this kind

$$\frac{\alpha x^m}{f^n + g^n x^n} + \frac{\beta x^m}{f^n + h^n x^n}$$

and it will be

$$\alpha f^n + \beta f^n = 1 \quad \text{and} \quad \alpha h^n + \beta g^n = 0,$$

whence it is

$$\alpha = \frac{1}{f^n} - \beta = -\frac{\beta g^n}{h^n}.$$

If the exponent m was greater than n, the transformation into the following fractions will be more convenient

$$\frac{\alpha x^{m-n}}{f^n + g^n x^n} + \frac{\beta x^{m-n}}{f^n + h^n x^n},$$

by means of which it is

$$\alpha + \beta = 0 \quad \text{and} \quad \alpha h^n + \beta g^n = 1$$

and hence

$$\alpha = \frac{1}{h^n - g^n} \quad \text{and} \quad \beta = \frac{1}{g^n - h^n}.$$

But no matter which of both transformations is used, both fractions to result this way will be resolved into its simple fractions, which taken together will be equal to the propounded fraction using the method explained before.

§420 In like manner the method treated up to this point will suffice, if the denominator consists of several terms of this kind

$$a + bx^n + cx^{2n} + dx^{3n} + ex^{4n} + \text{etc.},$$

if it only can be resolved into factors of this form $f^n \pm g^n x^n$. For, let us assume that this fraction is to be resolved into its simple factors

$$\frac{x^m}{(a - x^n)(b - x^n)(c - x^n)(d - x^n)\text{etc.}}.$$

At first, resolve it into these

$$\frac{Ax^m}{a - x^n} + \frac{Bx^m}{b - x^n} + \frac{Cx^m}{c - x^n} + \frac{dx^m}{d - x^n} + \text{etc.},$$

the numerators of which will be determined the following way that it is

$$A = \frac{1}{(b-a)(c-a)(d-a)\text{etc.}}$$
$$B = \frac{1}{(a-b)(c-b)(d-b)\text{etc.}}$$
$$C = \frac{1}{(a-c)(b-c)(d-c)\text{etc.}}$$
etc.

Therefore, after this preparation these single fractions will be resolved into their simple fractions by means of the method explained before.

§421 If a denominator of this kind

$$a + bx^n + cx^{2n} + dx^{3n} + \text{etc.}$$

has not only real factors of the form $f^n + \pm g^n x^n$, two imaginary ones are to be combined into a single real one. Therefore, let us put that the product of two factors of this kind is

$$f^{2n} - 2f^n g^n \cos\omega + g^{2n}x^{2n};$$

and because this expression has no simple real factors, let us put that the trinomial ones are contained in this general form

$$ff - 2fgx\cos\varphi + ggxx,$$

whose number will be $= n$. Therefore, having put $x^n = \frac{f^n}{g^n}\cos n\varphi$ this equation will result

$$1 - 2\cos\omega \cdot \cos n\varphi + \cos 2n\varphi = 0.$$

Furthermore, having put $x^n = \frac{f^n}{g^n}\sin n\varphi$ it will also be

$$-2\cos\omega \cdot \sin n\varphi + \sin 2n\varphi = 0,$$

which divided by $\sin n\varphi$ gives $\cos n\varphi = \cos\omega$ and so at the same time the first equation will be satisfied. Therefore, it will be $n\varphi = 2k\pi \pm \omega$ while k denotes any positive integer and hence it will be $\varphi = \frac{2k\pi\pm\omega}{n}$ and all factors will be contained in this form

$$ff-2fgx\cos\frac{2k\pi\pm\omega}{n}+ggxx,$$

whence one will have the following factors

$$\begin{aligned}
&ff-2fgx\cos\frac{\omega}{n}+ggxx\\
&ff-2fgx\cos\frac{2\pi-\omega}{n}+ggxx\\
&ff-2fgx\cos\frac{2\pi+\omega}{n}+ggxx\\
&ff-2fgx\cos\frac{4\pi-\omega}{n}+ggxx\\
&ff-2fgx\cos\frac{4\pi+\omega}{n}+ggxx\\
&\quad\text{etc.},
\end{aligned}$$

of which so many are to be taken until their number becomes $= n$.

§422 Therefore, if this fraction is propounded to be resolved into its simple fractions

$$\frac{x^{m-1}}{f^{2n}-2f^ng^n\cos\omega+g^{2n}x^{2n}},$$

since any trinomial factor of the denominator is contained in this form

$$ff-2fgx\cos\varphi+ggxx$$

while $\varphi=\frac{2k\pi\pm\omega}{n}$, consider this fraction

$$\frac{x^m}{f^{2n}x-2f^ng^nx^{n+1}\cos\omega+g^{2n}x^{2n+1}}$$

equal to it and put $x^m = P$ and the denominator

$$f^{2n}x-2f^ng^nx^{n+1}\cos\omega+g^{2n}x^{2n+1}=Q;$$

it will be

$$\frac{dQ}{dx}=f^{2n}-2(n+1)f^ng^nx^n\cos\omega+(2n+1)g^{2n}x^{2n}.$$

Therefore, by putting

$$x^n = \frac{f^n}{g^n} \cos n\varphi$$

it will be

$$\mathfrak{P} = \frac{f^m}{g^m} \cos m\varphi \quad \text{and} \quad \mathfrak{P} = \frac{f^m}{g^m} \cos \frac{m(2k\pi \pm \omega)}{n}$$

and

$$\mathfrak{Q} = f^{2n}(1 - 2(n+1)\cos\omega \cdot \cos n\varphi + (2n+1)\cos 2n\varphi).$$

But because it is $\cos n\varphi = \cos\omega$, it will be

$$\cos 2n\varphi = 2\cos^2\omega - 1$$

and hence

$$\mathfrak{Q} = f^{2n}(-2n + 2n\cos^2\omega) = -2nf^{2n}\sin^2\omega.$$

Further, having put

$$x^n = \frac{f^n}{g^n} \sin n\varphi$$

it will be

$$\mathfrak{p} = \frac{f^m}{g^m} \sin m\varphi = \frac{f^m}{g^m} \sin \frac{m(2k\pi \pm \omega)}{n}$$

and

$$\mathfrak{q} = -f^{2n}(2(n+1)\cos\omega \cdot \sin n\varphi - (2n+1)\sin 2n\varphi);$$

because of

$$\sin 2n\varphi = 2\sin n\varphi \cdot \cos n\varphi = 2\cos\omega \cdot \sin n\varphi$$

it will be

$$\mathfrak{q} = 2nf^{2n}\cos\omega \cdot \sin n\varphi.$$

But because it is $n\varphi = 2k\pi \pm \omega$, it will be $\sin n\varphi = \pm\sin\omega$ and

$$\mathfrak{q} = \pm 2nf^{2n} \sin\omega \cdot \cos\omega.$$

Having found these it will be

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = 4n^2 f^{4n} \sin^2\omega,$$

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{2nf^{m+2n}}{g^m}(\pm\cos m\varphi \cdot \sin\omega \cdot \cos\omega + \sin m\varphi \cdot \sin^2\omega)$$

or

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \pm\frac{2nf^{m+2n}}{g^m}\sin\omega \cdot \cos(m\varphi \mp \omega)$$

or

$$\begin{aligned}
\mathfrak{P}\mathfrak{q} \ - \mathfrak{p}\mathfrak{Q} &= \pm \ \frac{2nf^{m+2n}}{g^m}\sin\omega \cdot \cos\frac{2km\pi \pm (m-n)\omega}{n} \\
\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} &= \quad \frac{2nf^{m+2n}}{g^m}(-\cos m\varphi \cdot \sin^2\omega \pm \sin m\varphi \cdot \sin\omega\cos\omega), \\
\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} &= \pm \ \frac{2nf^{m+2n}}{g^m}\sin\omega \cdot \sin(m\varphi \mp \omega),
\end{aligned}$$

or

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{Q} = \pm \ \frac{2nf^{m+2n}}{g^m}\sin\omega \cdot \sin\frac{2km\pi \pm (m-n)\omega}{n}.$$

Therefore, from the factor of the denominator

$$ff - 2fgx\cos\frac{2k\pi \pm \omega}{n} + ggxx$$

this simple fraction results

$$\frac{\pm f\sin\frac{2k\pi\pm\omega}{n} \cdot \cos\frac{2km\pi\pm(m-n)\pi}{n} \pm \sin\frac{2km\pi\pm(m-n)\omega}{n}\left(gx - f\cos\frac{2k\pi\pm\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff - 2fgx\cos\frac{2k\pi\pm\omega}{n} + ggxx\right)}$$

or

$$\frac{gx\sin\frac{2km\pi\pm(m-n)\omega}{n} \pm f\sin\frac{2k(m-1)\pi\pm(m-n-1)\omega}{n}}{nf^{2n-m}g^{m-1}\sin\omega\left(ff - 2fgx\cos\frac{2k\pi\pm\omega}{n} + ggxx\right)}.$$

EXAMPLE

To resolve this fraction $\frac{x^{m-1}}{f^{2n}-2f^n g^n x^n \cos\omega + g^{2n}x^{2n}}$ *into its simple fractions.*

These simple fractions in question will therefore be

$$\frac{f\sin\frac{\omega}{n}\cdot\cos\frac{(m-n)\omega}{n}+\sin\frac{(m-n)\omega}{n}\left(gx-f\cos\frac{\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{\omega}{n}+ggxx\right)}$$

$$-\frac{f\sin\frac{2\pi-\omega}{n}\cos\frac{2m\pi-(m-n)\omega}{n}+\sin\frac{2m\pi-(m-n)\omega}{n}\left(gx-f\cos\frac{2\pi-\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2\pi-\omega}{n}+ggxx\right)}$$

$$+\frac{f\sin\frac{2\pi+\omega}{n}\cos\frac{2m\pi+(m-n)\omega}{n}+\sin\frac{2m\pi+(m-n)\omega}{n}\left(gx-f\cos\frac{2\pi+\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{2\pi+\omega}{n}+ggxx\right)}$$

$$-\frac{f\sin\frac{4\pi-\omega}{n}\cos\frac{4m\pi-(m-n)\omega}{n}+\sin\frac{4m\pi-(m-n)\omega}{n}\left(gx-f\cos\frac{4\pi-\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{4\pi-\omega}{n}+ggxx\right)}$$

$$+\frac{f\sin\frac{4\pi+\omega}{n}\cos\frac{4m\pi+(m-n)\omega}{n}+\sin\frac{4m\pi+(m-n)\omega}{n}\left(gx-f\cos\frac{4\pi+\omega}{n}\right)}{nf^{2n-m}g^{m-1}\sin\omega\left(ff-2fgx\cos\frac{4\pi+\omega}{n}+ggxx\right)}$$

etc.

and one has to continue this way until the number of these fractions was n. If m was a number either greater than $2n-1$ or negative, in the first case integral parts, in the second fraction are to be added which are easily found by means of the method explained before.